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QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

REPLIES.

21. For the diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{rrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation?[1] How may all the solutions of this equation be found by a systematic procedure?

Reply by E. B. Escott, Chicago, Ill.

A complete solution of questions of this kind does not seem to be possible. In default of a complete solution, I give four methods by which all of the given solutions may be found.

I. Let $x = a$, $y = b$ be a solution found by trial. From

$$x^2 - y^3 = 17, \tag{1}$$

and

$$a^2 - b^3 = 17, \tag{2}$$

we get

$$(x + a)(x - a) = (y - b)(y^2 + by + b^2). \tag{3}$$

Let $\dfrac{x - a}{y - b} = m$; or $x = m(y - b) + a$.

Substituting in (3), removing factor $y - b$ from both members and arranging,

$$y^2 - (m^2 - b)y + bm^2 - 2am + b^2 = 0. \tag{4}$$

The roots of (4) will be rational if its discriminant is a perfect square, i.e., if

$$m^4 - 6bm^2 + 8am - 3b^2 = r^2. \tag{5}$$

Solutions of this equation may be found by the following method, given by Euler in his *Elements of Algebra*, part II, Chapters 8 and 9.

For example, let $a = 3$, $b = -2$; then (5) becomes

$$m^4 + 12m^2 + 24m - 12 = r^2. \tag{6}$$

By trial, we find that $m = 1$ is a solution.

Let $m = p + 1$; then

$$p^4 + 4p^3 + 18p^2 + 52p + 25 = r^2. \tag{7}$$

Let

$$r = ep^2 + fp + g.$$

Substituting in (7) and combining terms

$$(e^2 - 1)p^4 + (2ef - 4)p^3 + (2eg + f^2 - 18)p^2 + (2fg - 52)p + (g^2 - 25) = 0. \tag{8}$$

In (8) we may choose e, f, and g so that the equation reduces to one of the first degree in p.

(*a*) If $e^2 - 1 = 0$, $2ef - 4 = 0$, $2eg + f^2 - 18 = 0$, we have $e = 1$, $f = 2$, $g = 7$.

Then

$$p = 1, \text{ and } m = 2.$$

(*b*) If $e^2 - 1 = 0$, $2ef - 4 = 0$, $g^2 - 25 = 0$, we have $e = 1$, $f = 2$, $g = \pm 5$.

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[1] In *Mathematical Questions and Solutions from the "Educational Times,"* new series, Vol. 8, 1905, p. 53, R. F. Davis gives all of the solutions indicated above and purports to show that "any single solution leads to an interminable chain of other solutions." The question is also discussed by A. Cunningham, *Mathematical Questions*, etc., *l.c.*, p. 54; and vol. 14, 1908, pp. 107–108. See also *L'Interméd. des Math.*, 1905–13, Qu. 2512, 4057.—Editor-in-Chief.

Then

$$p = -8, \text{ and } -3; \text{ and } m = -7, \text{ and } -2.$$

(*c*) If $e^2 - 1 = 0$, $2fg - 52 = 0$, $g^2 - 25 = 0$, we have $e = 1$, $f = \pm 26/5$, $g = \pm 5$.
Then

$$p = -119/40, \text{ and } -1/15; \text{ and } m = -79/40, \text{ and } 14/15.$$

(*d*) If $2eg + f^2 - 18 = 0$, $2fg - 52 = 0$, $g^2 - 25 = 0$, we have $e = -113/125$, $f = 26/5$, $g = 5$.

Then $p = -8725/119$, and $m = -8606/119$.

Substituting these values of m in (4), we get the following solutions:

m	y		x	
2	8	-2	23	3
-7	52	-1	-375	-4
-2	4	2	-9	-5
$-\frac{79}{40}$	$\frac{137}{64}$	$\frac{94}{25}$	$-\frac{2651}{512}$	$-\frac{1047}{125}$
$\frac{14}{15}$	$-\frac{8}{9}$	$\frac{94}{25}$	$\frac{109}{27}$	$\frac{1047}{125}$

Since one solution just found is $x = 5$, $y = 2$, we can take $a = 5$, $b = 2$.
Substitute these values in (5).

$$m^4 - 12m^2 + 40m - 12 = r^2. \tag{9}$$

Since $m = 2$ is a solution, let $m = p + 2$ and proceed as before. We find the following values of m, and the corresponding solutions:

m	y		x	
2	4	-2	9	-3
3	8	-1	23	-4
-7	4	43	-9	-282

Similarly, putting $a = 375$, $b = 52$, we find $m = 73$; whence $y = 43$, and 5234;
$x = -282$, and 378661.

The above method will give an indefinite number of rational solutions but apparently only a limited humber of integral solutions.

II. Consider the curve of the third degree,

$$x^2 - y^3 = 17. \tag{1}$$

The tangent at (a, b) is

$$2ax - 3b^2y + a^2 - 51 = 0. \tag{2}$$

This cuts the curve again in the point

$$x = -\frac{(a^2 + 153)^2 - 31212}{8a^3}, \quad y = \frac{b}{4a^2}(a^2 - 153). \tag{3}$$

Taking any rational solution (a, b) of (1), we get a second rational solution from (3). From this solution we get another, etc.

Example. From $a = 3$, $b = -2$, we get from (3)

$$x = 23, \quad y = 8.$$

III. Under the first method, equation (4) is a quadratic equation in y. Calling the roots y_1 and y_2 we have

$$y_1 + y_2 = m^2 - b.$$

If we have for a given value of m, one value of y, the other value may be found from this relation.

Considering the same equation as a quadratic in m, and calling the roots m_1 and m_2 we have

$$m_1 + m_2 = -\frac{2a}{y - b}.$$

In the same way given a value of y and one value of m, we can find another value of m.

In this way we can find very rapidly a chain of values of m and y.

Example: Let $a = 3$, $b = -2$.

$$y_1 + y_2 = m^2 + 2, \tag{1}$$

$$m_1 + m_2 = -\frac{6}{y+2}. \tag{2}$$

$$\begin{array}{llll}
\text{Starting with} & y = -2, & m = 2, & (1) \text{ gives } y = 8, \\
 & & & (2) \text{ gives } m = \infty; \\
 & y = 8, & m = 2, & (2) \text{ gives } m = -\tfrac{13}{5}; \\
 & y = 8, & m = -\tfrac{13}{5}, & (1) \text{ gives } y = \tfrac{19}{25}; \\
 & y = \tfrac{19}{25}, & m = -\tfrac{13}{5}, & (2) \text{ gives } m = -\tfrac{49}{115}; \\
 & \cdots & \cdots & \cdots
\end{array}$$

IV. Solutions of $x^2 - y^3 = 17$ in integers may be found by Gauss's "Method of Exclusion" by which *all* the solutions under any desired limit may be found.

Since
$$y^3 + 17 = x^2,$$
the residues of $y^3 + 17$ to any modulus must be quadratic residues. Therefore, we can exclude all values of y for which $y^3 + 17 \equiv$ a non-quadratic residue. This gives us the following:

$y \not\equiv 0 \pmod{3}$;
$y \not\equiv 1, 5 \pmod{8}$, $y \not\equiv 7 \pmod{8}$, excepting $y \equiv 15 \pmod{16}$;
$y \not\equiv 0, 1 \pmod{5}$, $y \not\equiv 2 \pmod{5}$, excepting $y \equiv 2 \pmod{25}$;
$y \not\equiv 0 \pmod{7}$;
$y \not\equiv 0, 1, 5, 6, 7 \pmod{11}$, $y \not\equiv 3 \pmod{11}$, excepting $y \equiv -30 \pmod{121}$;
$y \not\equiv 1, 3, 9 \pmod{13}$;
$y \not\equiv 0, 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$;
$y \not\equiv 1, 7, 10, 11, 13, 15 \pmod{19}$;

. . . .

Rejecting all the cases above and others (mod 23 and 29) we find that the only possible solutions less than 1,000 are
$$y = 2, 4, 8, 43, 52, 934, 988.$$

Trying these values, we find the first five give solutions. The last two do not.

Remark. It is worth while noting the essential difference between the equations
$$x^2 - a = y^3,$$
and
$$x^2 + a = y^3.$$

The latter equation has only a finite number of solutions, which can be found by a perfectly general method,[1] but it is quite different with the first equation. The reason is that the properties of quadratic forms with positive determinant are quite different from those with a negative determinant. (The word "determinant" is used in its original sense of discriminant.)

NOTE. In the last section of an article[2] entitled "The Diophantine Equation $y^2 - k = x^3$," L. J. Mordell indicates the results obtained as to the existence of an infinite number of solutions, a finite number, or none, for all values of k between ± 100. In accordance with his list, the case $k = 17$ should yield an infinite set of integral solutions. The question under discussion thus still presents as a challenge to our readers the alternative of either disproving Mr. Mordell's result for the case $k = 17$, or else verifying it and producing one or more new solutions.—EDITOR.

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[1] Gerono showed (*Nouvelles Annales de Mathématiques*, 1877, pp. 325–326) that there are *no* integral solutions of the equation $x^2 + 17 = y^3$.—Editor-in-Chief.

[2] *Proceedings of the London Mathematical Society*, series 2, vol. 13 (1914), pp. 60–80.

32. In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$(1)\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\ (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\ (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\ (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\ x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\ x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\ x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating x_1, y_1, x_2, y_2, x_3, y_3 gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
   (*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

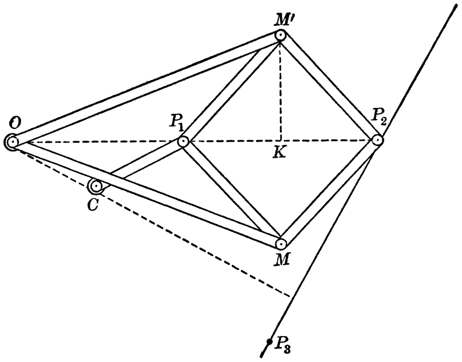
$$(8)\ f_1(x_1, y_1) = 0.$$

   (*b*) From equations (2), (4), (6) eliminate x_3 and y_3 and obtain an equation

$$(9)\ f_2(x_1, y_1) = 0.$$

   (*c*) From equations (7), (8), (9) eliminate x_1 and y_1 and obtain the desired equation.
2. How should this procedure be supplemented to secure the result?

Reply by G. H. Ling, University of Saskatchewan.



The configuration discussed has a line of symmetry; the equations developed show that the points (x_2, y_2) and (x_3, y_3) are *either coincident or symmetrical with respect to the line of symmetry*, but they fail to express that the essential feature of the configuration is that the two points mentioned are *not* coincident.

The suggested procedure for elimination cannot take account of this unexpressed fact. Equations (8) and (9) are identical and the elimination of x_1 and y_1 from (7), (8) and (9) halts because of the lack of three independent equations.

The procedure may be supplemented as follows:

If two of the three equations (1), (3), (5) be solved for x_2 and y_2 two distinct solutions are obtained and these must be the values for x_2 and y_2, and for x_3 and y_3. Both of these sets of values must satisfy that one of the equations (1), (3), (5) which is not employed in the solution just mentioned. The substitution of both of these sets of values in this third equation yields two equations (8) and (9) which associated with the equation (7) yield a single resultant equation free from x_1 and y_1. This last equation is the equation of the desired locus. (See May Monthly, p. 188.)

It is easy to see that the case here discussed can be generalized, though in the more general cases the treatment of the problem would not be nearly so simple.

DISCUSSIONS.

Every teacher of trigonometry, analytic geometry, and the calculus has at times experienced the difficulty of inducing students to regard the radian measure of an angle as a pure number. How is this difficulty to be met? Professor Car-

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[1] If reference is made to the article on "Linkages" in the December, 1915, Monthly, by Mr. Leavens, the following coördinates may be applied to his figure (reproduced above): $O\ (0, 0)$; $C\ (c, 0)$; $P_1\ (x_1, y_1)$; $M(x_2, y_2)$; $M'\ (x_3, y_3)$; $P_2\ (X, Y)$.